

On Arguments for Linear Quantum Dynamics

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Abstract

Two recent arguments for linear dynamics in quantum theory are critically re-examined. Neither argument is found to be satisfactory as it stands, although an improved version of one of the arguments can in fact be given. This improved version turns out to be still not completely unproblematic, but it is argued that it contains only a single actual loophole, which is identical to a loophole that remains in experimental proofs of nonlocality of Bell-type. It is concluded that - within the context of the standard quantum kinematical framework and in agreement with what has been concluded by earlier authors - a nonlinear dynamics of density operators is inconsistent with relativistic causality. However, it is also stressed that this conclusion in itself has little implication for the nature of dynamics at the Hilbert space level - in particular, it does not force dynamics to be linear at this level - nor does it continue to be valid in contexts that go beyond the standard quantum kinematical framework. Despite their seeming triviality, these last two points have not always been appreciated in the literature. Finally, it is also pointed out that the argument for complete positivity, as given in conjunction with one of the two recent arguments for linear dynamics, in fact only establishes a condition that is weaker than complete positivity.

1 Introduction

Given the standard kinematical framework for quantum theory (or the “minimal” extension of this framework¹), it is natural to ask whether it’s possible to consistently generalize the standard unitary dynamics of the theory, or whether perhaps this dynamics is already fixed by the kinematics, together with certain other, physically reasonable assumptions.

To this end it is recalled that in ordinary quantum theory, as well as in classical mechanics, dynamical evolution is usually viewed as being generated by a preferred observable, i.e. the Hamiltonian. Thus, within the context of the standard quantum kinematical framework (or its minimal extension), in which the Hamiltonian is represented by a self-adjoint linear operator, it seems at first that dynamics can only be unitary. Yet, an important reason for considering non-standard dynamical evolution laws for quantum theory comes from the existence of the projection postulate, which is assumed in most textbook formulations of the theory and which appears to introduce a second kind of dynamics into it. One is thus forced to either reject this postulate, in which case it becomes necessary to work out a specific, convincing interpretation of quantum theory, in which this postulate is not needed (e.g. a nonlocal hidden variable interpretation), or to accept it, in which case overall conceptual consistency would seem to require a demonstration that both types of dynamics can be viewed as different limits of a single dynamics, that arises within the

¹The standard way of representing states and observables in quantum theory, together with the squared modulus rule for calculating probabilities - or, equivalently, the trace rule - is here referred to as the (standard) kinematical framework for quantum theory. It is sometimes convenient to slightly generalize this standard framework, by allowing states to be represented by general density operators, while keeping the rest of the framework fixed. The resulting structure will occasionally be referred to as the minimally extended kinematical framework.

context of a deeper underlying theory². If the second route is followed, an important reason for (initially) restricting attention to generalizations of dynamics, rather than also admitting, say, nonlinear observables and state spaces more general than (projective) Hilbert spaces, is that it appears to be much more difficult to generalize the kinematics, while still allowing for a consistent physical interpretation of such a generalized framework. A framework that incorporates standard quantum kinematics but in which the dynamics is generalized, could then be regarded as a “lowest order” nontrivial approximation to a deeper nonlinear theory. But, in pursuing such a strategy, it would be very useful to know if the standard kinematical framework (together with certain other, physically reasonable assumptions) imposes any constraints on the nature of the dynamics. It is the purpose of this note to critically re-examine two recent arguments for the linearity of quantum dynamics in this regard. The first argument is basically that relativistic causality requires quantum dynamics to be linear [1]. It involves spatially separated, but entangled systems and is a different version of an older, essentially identical argument [2, 3]. The improvement with respect to the earlier version of the argument is argued to lie in the supposed facts that the projection postulate need not be assumed to establish the argument and that a certain condition, known as “complete positivity”, is implied by the properties of the dynamics. It will be seen however that, although it may be possible to establish the argument without using the projection postulate, additional assumptions would then have to be made. In any case, without the projection postulate, the linearity of dynamics does not merely follow - as contended in Ref. [1] - from quantum kinematics and relativistic causality³. It will also be pointed out furthermore, that the remainder of the argument actually establishes a condition weaker than complete positivity. The second recent argument for linear quantum dynamics mentioned above, is basically that if time-evolution is implemented by a well-defined map on the space of density operators, such that all possible ways of “experimentally realizing” an application of the map to a given density operator always yield the same result, then this map must be linear [5]. No entanglement is used in this argument and it does not suffer from a loophole that remains in the argument based on relativity. However, the statement that the assumed property of the map implies its linearity, is almost a trivial assertion, whereas it will be argued that there are no good reasons (apart from causality) to require that a map implementing time-evolution on a system should in fact satisfy this property.

2 Entangled Systems : Linear Dynamics and Causality

Consider a composite system, described in terms of the tensor product, $\mathcal{H}_I \otimes \mathcal{H}_{II}$, where the Hilbert spaces \mathcal{H}_I and \mathcal{H}_{II} are assumed to be finite-dimensional and are thought of as supporting states in causally disconnected regions of spacetime, \mathcal{O}_I , \mathcal{O}_{II} . It is furthermore assumed that the total system has been prepared in some pure entangled state, $|\Psi\rangle$, that is taken to be normalized. Physical conditions in \mathcal{O}_I are completely described by the reduced density operator, $\rho_I = \text{Tr}_{II} \rho$, $\rho := |\Psi\rangle\langle\Psi|$, but it is important to note that any statistical mixture associated with this density operator is an improper mixture in the present context⁴ [7]. In other words, for any ρ_I -ensemble, $\{w_i, P_{\psi_i}\}$, one cannot consistently interpret ρ_I as describing a large ensemble of physical systems, such that a fraction w_i of these systems are in the exact quantum state $|\psi_i\rangle$ with certainty. Consequently, with this interpretation of ρ_I , the only object of physical relevance on which an operator, $\$,$ representing dynamical evolution in \mathcal{O}_I , can act, is ρ_I itself. In particular, it is physically meaningless to ask how the individual projectors, P_{ψ_i} , of a particular ρ_I -ensemble are affected by the dynamics. The only consistency requirement that follows from this is that if T denotes the operator implementing dynamical evolution on the composite system (where both T

²Alternatively, projections could be attributed to “free choices” made by the Universe, that are not associated with an underlying equation of motion - e.g. a stochastic modification of the Schrödinger equation. A third option is discussed in section 2, but in this case severe interpretational difficulties arise.

³A somewhat similar criticism has been raised before [4].

⁴It is recalled that any density operator, ρ , has infinitely many ρ -ensembles associated with it, i.e. statistical mixtures, $\{w_i, P_{\psi_i}\}$, such that $\rho = \sum_i w_i P_{\psi_i}$, where $P_{\psi_i} = |\psi_i\rangle\langle\psi_i|$, $\|\psi_i\| = 1$ (the states $|\psi_i\rangle$ are not necessarily orthogonal) and $0 < w_i < 1$, $\sum_i w_i = 1$ [6] (in the pure case, all of these ensembles are of course trivial; i.e. $P_{\psi_i} = P_\psi$, for some fixed state $|\psi\rangle$, for all i).

and $\$$ are imagined to refer to the same fixed time-interval in some particular reference frame), one must have $\$\rho_I = \text{Tr}_{II} \mathbb{T} \rho$.

It is an important mathematical fact, that for any ρ_I -ensemble, $\{w_i, P_{\psi_i}\}$, it is always possible (after enlarging the Hilbert space \mathcal{H}_{II} to another finite-dimensional Hilbert space, \mathcal{H}'_{II} , if necessary), to express the state $|\Psi\rangle$ as

$$|\Psi\rangle = \sum_i \sqrt{w_i} |\psi_i\rangle |\phi_i\rangle \quad (1)$$

where the $|\phi_i\rangle$ constitute an orthonormal basis for \mathcal{H}_{II} [6]. If the projection postulate is assumed, this physically means that the ρ_I -ensemble $\{w_i, P_{\psi_i}\}$ can be generated by local operations in \mathcal{O}_{II} (i.e. after first introducing an extra local quantum system in some standard state, a so-called ancilla, if necessary, to enlarge the original Hilbert space \mathcal{H}_{II} [8], and then perform a measurement in the basis $\{|\phi_i\rangle\}$) and that this ρ_I -ensemble now describes a statistical mixture which is proper (so that it is possible [7] to consistently assign an ignorance interpretation to $\{w_i, P_{\psi_i}\}$). In other words, if the projection postulate is assumed, any (proper) ρ_I -ensemble in \mathcal{O}_I can be generated by local measurements in \mathcal{O}_{II} . Now, let $\$$ again denote the operator implementing dynamical evolution in \mathcal{O}_I and assume this operator to be defined initially on pure states only. Since the time-interval associated with $\$$ is arbitrary, one must have

$$\sum_i w_i \$P_{\psi_i} = \sum_i \tilde{w}_i \$P_{\tilde{\psi}_i} \quad (2)$$

for any two ρ_I -ensembles $\{w_i, P_{\psi_i}\}$ and $\{\tilde{w}_i, P_{\tilde{\psi}_i}\}$, as otherwise it would be possible to communicate at superluminal speeds. Since $\{w_i, P_{\psi_i}\}$, $\{\tilde{w}_i, P_{\tilde{\psi}_i}\}$ and ρ_I are arbitrary, this means that $\$$ can be extended to a map, also denoted by $\$$, on the space, $\Xi(\mathcal{H}_I)$, of density operators on \mathcal{H}_I , which is linear in the sense that

$$\$\rho_I = \$ \sum_i w_i P_{\psi_i} = \sum_i w_i \$P_{\psi_i} \quad (3)$$

for any $\rho_I \in \Xi(\mathcal{H}_I)$ and any ρ_I -ensemble $\{w_i, P_{\psi_i}\}$ (alternatively, it could have been assumed that $\$$ is defined on $\Xi(\mathcal{H}_I)$ to begin with, so that the left-hand side of Eq. (3) represents the time-evolved state in \mathcal{O}_I in the case where no measurement has been performed in \mathcal{O}_{II} ; the fact that the state in \mathcal{O}_I should be independent of the conditions in \mathcal{O}_{II} then establishes Eq. (3)).

Both earlier arguments for linear quantum dynamics [2, 3] have the above general structure of exploiting EPR-correlations to the extent that different descriptions of the same state in \mathcal{O}_I , that exist as a result of what has happened in \mathcal{O}_{II} , must dynamically evolve into the same state to avoid superluminal signalling. These arguments assume the projection postulate, either implicitly or explicitly. In the more recent version of the argument [1] this postulate is not assumed, but other than that, the argument is identical to that given here. It will now be seen however, that without the projection postulate, the argument is inconclusive. The crucial point of the previous discussion is that in order to establish the linearity of $\$$, it is necessary to assume that the act of carrying out a measurement in \mathcal{O}_{II} induces physical effects in \mathcal{O}_I , in such a way that the interpretation of a specific ρ_I -ensemble (associated with a particular orthonormal basis for \mathcal{H}_{II} , for a given total state $|\Psi\rangle$) undergoes a transition from an improper to a proper mixture (and it is rather clear that such a transition must correspond to something physical, as the two types of mixture refer to distinct experimental situations). But, in terms of the mathematical formalism, it is not clear how to incorporate this assumption without assuming that some appropriate form of the projection postulate holds true, unless some specific further assumptions are made with regards to *interpreting* the mathematical formalism. To see this, the standard von Neumann account of measurements - which is assumed explicitly in the argument of Ref. [1] - is briefly reviewed first, in order to determine the result of carrying out a measurement in \mathcal{O}_{II} . If the measurement apparatus is treated quantum mechanically, the result of an idealized measurement in \mathcal{O}_{II} according to von Neumann is the state

$$|\tilde{\Psi}\rangle := \sum_i \sqrt{w_i} |\psi_i\rangle |\phi_i\rangle |A_i\rangle \quad (4)$$

where the apparatus states, $|A_i\rangle$, represent definite “pointer readings” corresponding to a measurement in the basis $|\phi_i\rangle$. Von Neumann’s point was that if everything is to be treated in quantum mechanical terms, one could insert arbitrarily long, finite chains of further physical systems (i.e. other measurement apparatuses, sensory and nervous systems of “observers”, etc.), with which the original system plus measurement apparatus becomes entangled and that, in order to account for definite observational facts, it ultimately does not matter exactly *where* in this chain projections are introduced. Of course, the problem of how to account for definite observational facts within quantum theory - i.e. the “measurement paradox” - and in particular, the issue of whether it is necessary to introduce projections to resolve this paradox, has been the source of much controversy and some remarks are in order here. However, first note that from the viewpoint of the general discussion, the assumption of von Neumann measurements is somewhat odd, since such a notion of measurements entails the assumption of a linear - in fact unitary - dynamical evolution of the total system before and during the measurement process. But the implication of considering the possibility of a generalized dynamics is that there could be physical conditions for which this notion of measurements is inadequate, in which case it is not clear that the argument for linear dynamics does not break down. However, this difficulty can be circumvented simply by assuming the projection postulate, without making any assumptions about the actual measurement interactions. But if this point is ignored, the problem from the viewpoint of the general discussion is whether proper mixtures can be generated in \mathcal{O}_I , as a result of carrying out von Neumann measurements in \mathcal{O}_{II} . Now, the logical inference of von Neumann’s account of measurements - and it is again emphasized that this account is based on the assumption that the dynamics of the total system remains unitary, as the degrees of freedom of the system of interest become entangled with increasingly more degrees of freedom of other systems in the chain - strongly seems to be that *if* it is believed necessary to introduce projections to account for definite observational facts, the only reasonable place to do so would be the point where the “consciousness” of an “observer” enters. Although such an idea has indeed been taken seriously by a number of physicists, most notably von Neumann himself and Wigner, it is easily seen to lead to a number of serious difficulties. An immediate mathematical problem is the fact that the argument *assumes* from the outset that only “classically reasonable” states are ultimately allowed to occur or be “experienced”, or, equivalently, it assumes that there are preferred bases in Hilbert space⁵, something which has no natural place in the mathematical formalism of quantum theory and therefore needs further justification. Far more problematic however, are the conceptual implications of the view that it is the act of conscious observation that reduces the quantum state. As has been stressed by a number of authors (see e.g. Ref. [9]), it is far from clear what “consciousness” precisely is and what amount of it would furthermore be required to effect state reduction. In addition, it has been pointed out that the view “does not respect the symmetry that the facts are invariant under interchange of observers” [10], constituting the so-called concordance problem. In other words, the upshot of following von Neumann’s account of measurements all the way, is that it leads to a view very similar to solipsism.

The reason for reviewing von Neumann’s account of the measurement process - of which projections are a part - is that it is easy to shift from the solipsist view, in which the quantum state is “objectively” reduced by a single observer’s consciousness, to the opposite extreme, in which all alternatives continue to exist in parallel and the state reduction only takes places subjectively - in the mind of the observer. That is, it seems that if one follows von Neumann’s account of measurements all the way, one may just as well dispose of projections altogether and adopt an essentially Many Worlds type of perspective⁶. Is this what is implicitly assumed in the argument

⁵The Schmidt polar decomposition - given by Eq. (1) in the case where the w_i and $|\psi_i\rangle$ are respectively the nonzero eigenvalues and eigenstates of ρ_I - does not correspond to a preferred basis for $\mathcal{H}_I \otimes \mathcal{H}_{II}$ in this regard, since the eigenstates of ρ_I are not in general “classically reasonable”. However, the idea of “environment induced decoherence” leads, under certain assumptions, to a preferred basis of apparatus states, $|A_i\rangle$, a so-called “pointer basis”. See the main text for further discussion.

⁶This characterization of the Many Worlds viewpoint is intended to be flexible. It may either refer to a literal perspective, with “copies” of a single “observer” in different states of perception co-existing in different worlds, or it may refer to a “Many Minds” perspective, in which there is only a single world, but with a “branching of minds”, reflecting the different states of perception.

of Ref. [1] ? Although it strongly appears that such a view is logically implied by employing von Neumann's account of measurements without using projections, it may be objected that there are a number of "no-collapse" interpretations [11], which differ with respect to specific details. But the argument of Ref. [1] does not even mention any particular such interpretation and it is far from clear that the argument is actually interpretation independent - within the class of such no-collapse interpretations. Furthermore, contrary to what has sometimes been claimed in the literature (see e.g. Ref. [12]), decoherence is actually of no help as far as generating definite observational facts is concerned. For what the theory of "environment induced decoherence" [13] merely shows, is that upon modeling the external environment in an appropriate way and inserting it in the von Neumann chain, quantum interference effects quickly wash out, as a result of the system plus measurement apparatus interacting with the external environment. More precisely, what the theory shows is that, under appropriate conditions, the reduced density operator, $\rho_{I,II,M_{II}}$, associated with the system and measurement apparatus, rapidly becomes diagonal with respect to the specific "pointer basis" of apparatus states, $|A_i\rangle$, i.e. $\rho_{I,II,M_{II}} = \sum_i w_i P_{\psi_i} \otimes P_{\phi_i} \otimes P_{A_i}$. But this only shows that $\rho_{I,II,M_{II}}$ behaves as a probabilistic mixture. To say that a particular probabilistic mixture is actually realized as a result of the measurement process - as necessary for the linearity argument - requires adopting a specific view on how the quantum state at the Hilbert space level, which is still a superposition of macroscopically distinct states, i.e. Eq. (4), is to be interpreted⁷. If one insists on not using the projection postulate, there are several possible views that can be taken, as mentioned above. One may then attempt to establish the argument for linear dynamics within a specific such view, but the validity of the argument would then depend on the validity of the specific view adopted. For instance, in the case of a Many Worlds type of interpretation, it is not clear that outcomes obeying the square modulus rule can be shown to arise as "typical" by postulating an appropriate measure on the world Hilbert space. Even when decoherence effects are taken into account, the state represented by Eq. (4) can still be decomposed in infinitely many *different* ways and these must all have measure zero⁸. More generally, if one insists on an entirely linear, orthodox quantum treatment of all physical systems, it is necessary to explain how the environment enters a sufficiently "classical state", necessary to trigger decoherence in the first place. From a more fundamental viewpoint, this merely shifts the whole problem of *even accounting for the appearance of classical behaviour*. It is of course true that interpretational models in which state reduction is viewed as a real physical phenomenon also encounter some serious difficulties. However, this does not affect the argument for linear dynamics, as projections may simply be introduced on an ad hoc basis - although they are of course motivated by the idea that state reduction is objective. Furthermore, as already pointed out, there is something intrinsically ill-motivated about the entire argument if the standard von Neumann account of measurements is assumed to apply universally.

The upshot of the previous discussion is that the argument for linear dynamics, as presented in this section, is not conclusive if the projection postulate is not assumed. On the other hand, within the von Neumann context of measurements, it is clear from Eq. (4) that if the apparatus was observed in the particular state $|A_k\rangle$, say, the state of the system in \mathcal{O}_I is $|\psi_k\rangle$, if it is assumed that immediately after the measurement was performed, the state $|\tilde{\Psi}\rangle$ is reduced to $(\mathbb{1} \otimes \mathbb{1} \otimes P_{A_k})|\tilde{\Psi}\rangle = \sqrt{w_k}|\psi_k\rangle|\phi_k\rangle|A_k\rangle$. This option is explicitly rejected in Ref. [17], because it is pointed out that it could happen that the system in \mathcal{O}_{II} is destroyed by the measurement, something which would effectively modify the Hilbert space of the system according to $\mathcal{H}_I \otimes \mathcal{H}_{II} \otimes \mathcal{H}_M \rightarrow \mathcal{H}_I \otimes \mathcal{H}_M$ (with \mathcal{H}_M denoting the Hilbert space of the measurement apparatus). Instead of Eq. (4) one would then have

$$|\Psi\rangle \longrightarrow |\tilde{\Psi}\rangle := \sum_i \sqrt{w_i}|\psi_i\rangle|A_i\rangle \quad (5)$$

⁷For further discussion of this point, see e.g. Ref. [14].

⁸For critical reviews of the Many Worlds programme in general, see Ref. [15] and further references therein. For critical discussions concerning the status of proposed derivations of the square modulus rule within this programme, see Ref. [16].

However, in this particular case, even though one cannot use the projection postulate on $\mathcal{H}_I \otimes \mathcal{H}_{II}$ (as there is no projection operator from $\mathcal{H}_I \otimes \mathcal{H}_{II}$ to \mathcal{H}_I), it is still possible to use the postulate on $\mathcal{H}_I \otimes \mathcal{H}_M$, or a larger Hilbert space. Thus, by assuming (some appropriate form of) the projection postulate, it becomes possible to generate proper mixtures at a distance and the linearity of dynamics can be established. According to the argument in Ref. [1], it is merely the trace rule that “leads to the preparation at a distance of probabilistic mixtures”. More precisely, it is argued that the state in \mathcal{O}_I can be deduced by the observer in \mathcal{O}_{II} , by calculating the conditional probability for every projection operator for the system in \mathcal{O}_I , given a definite measurement outcome in \mathcal{O}_{II} . If it is again supposed that the total state is given by Eq. (1) and that the observer in \mathcal{O}_{II} carries out a measurement in the orthonormal basis $\{|\phi_i\rangle\}$, the probability to find the system in \mathcal{O}_I in the state P_{ψ_j} , conditional upon the system in \mathcal{O}_{II} having been found in the state P_{ϕ_i} , is just $\text{Tr}\rho(P_{\psi_j} \otimes P_{\phi_i})/\text{Tr}\rho(\mathbb{1} \otimes P_{\phi_i}) = \text{Tr}P_{\psi_i}P_{\psi_j}$ and according to Ref. [1], the state of the first system in the present context is given by $\text{Tr}_{II}\rho(\mathbb{1} \otimes P_{\phi_i})/\text{Tr}\rho(\mathbb{1} \otimes P_{\phi_i}) = P_{\psi_i}$. Thus, what this argument actually appears to amount to, is just the usual plausibility argument for the projection postulate; if a measurement of P_{ψ_i} on the \mathcal{O}_I -system is certain to give the eigenvalue 1, then the state of this system actually *is* the eigenstate P_{ψ_i} . In other words, the authors of Ref. [1] appear to claim that they are able to *derive* the projection postulate from the kinematical framework of quantum theory. But, in view of the existence of no-collapse interpretations, such a claim would clearly be contentious.

3 Further Discussion

To summarize the discussion of the previous section, it was argued - in concordance with earlier arguments - that the absence of superluminal signalling, i.e. relativistic causality, together with the standard kinematical framework for quantum theory and (some appropriate form of) the projection postulate, imply that the dynamics of density operators is necessarily linear and that, furthermore, in spite of what has been claimed in the literature, the projection postulate assumption has so far not been demonstrated to be dispensable for this argument. In this section, we address two types of question which are naturally raised by the above argument :

- (i) Is the argument rigorous ? If loopholes remain, how plausible are they ?
- (ii) Given that dynamical evolution on the space of density operators is linear, what does this imply for the nature of dynamical evolution on the corresponding Hilbert space ? What does it imply for the nature of dynamics if the corresponding Hilbert space is a factor in a tensor product ?

Concerning the first type of question, it is not difficult to see that the paradoxical nature of preparing ensembles at a distance points to a loophole in the argument [5, 18, 19]. Given any subset, S , of Minkowski spacetime, let $I^+(S)$, $J^+(S)$, denote respectively the “chronological future” and the “causal future” of S (i.e. $I^+(S)$, resp. $J^+(S)$, is the set of all points in Minkowski spacetime that can be joined to a point in S by a future directed curve, which is timelike, resp. null or timelike, and which starts in S ; cf. Ref. [20]). Let y furthermore denote the event in \mathcal{O}_{II} at which a measurement is carried out on the second system. The loophole is that the state of the first system reduces *causally* as a result of this measurement. That is, if the second system was found in the state P_{ϕ_j} , the state of the first system changes from ρ_I to $\text{Tr}_{II}\rho(\mathbb{1} \otimes P_{\phi_j})/\text{Tr}\rho(\mathbb{1} \otimes P_{\phi_j}) = P_{\psi_j}$, as the worldline in $I^+(\mathcal{O}_I)$ of the first system enters $J^+(y)$ (in Ref. [18] the notion of a “local state” of a system is introduced, in terms of a limiting procedure of a sequence of spacelike hypersurfaces that tend to the past lightcone of the system’s instantaneous spacetime location, thereby allowing the system’s state to change in a causal manner, as a result of performing measurements on a second system with which the system is entangled). Upon extending this construction to ensembles of entangled systems, it is thus seen that proper mixtures are generated in a causal manner in $I^+(\mathcal{O}_I) \cap J^+(\mathcal{O}_{II})$, as a result of performing measurements in \mathcal{O}_{II} . But it is clear that there is now no linearity restriction on the dynamical evolution of the first system. Thus, if state reduction is

assumed to occur in a causal fashion, a nonlinear dynamics of density operators is consistent with causality. It is possible to address this view on state reduction by experiment. If Δ denotes the minimum invariant distance between \mathcal{O}_I and \mathcal{O}_{II} , it takes at least a time Δ , according to this view, for the mixture in $I^+(\mathcal{O}_I)$ to become proper. Thus, by letting the experimenters in \mathcal{O}_I and \mathcal{O}_{II} make prior arrangements on what measurements to perform and by choosing Δ large enough, it is possible for the experimenters to determine - after comparing their measurement results - whether the mixture in \mathcal{O}_I was indeed improper for at least a time Δ . In order to definitely close this loophole in the argument, it is estimated that Δ should be of the order of $3 \cdot 10^4 \text{ km} \simeq 0.1 \text{ lightsec}$ [21]. However, although it seems to be consistent with present experimental facts to think that state reduction occurs in a causal fashion, it is difficult to believe that this corresponds to the actual state of affairs in Nature, as it would entail a radical departure from what is currently seen in Bell-type experiments as soon as the measurement events in such experiments can definitely be regarded as spacelike separated⁹. The problem with this strategy more generally, is that it will never be possible to close *all* loopholes completely in Bell-type experiments to everyone's satisfaction¹⁰ (see also the remarks below). Nevertheless, with respect to the specific loophole discussed here, it is reasonable to require that experiments address it in due time (and to moreover expect that, when this is done, the loophole will be closed).

The second type of question listed above concerns the ramifications of the linearity of time-evolution of (reduced) density operators. It is clear that a linear time-evolution on a Hilbert space, \mathcal{H} , implies a linear time-evolution on the corresponding space of density operators, $\Xi(\mathcal{H})$. Conversely however, a linear time-evolution on $\Xi(\mathcal{H})$ - again denoted by $\$$ - in itself implies very little for the nature of time-evolution at the Hilbert space level. If $\$$ is invertible, it maps pure states into pure states and it is not hard to show that this implies that $\$$ preserves the absolute values of inner products [5]. Hence, by Wigner's theorem, the action of $\$$ then takes the form

$$\$ \rho = U \rho U^\dagger \quad (6)$$

with $U : \mathcal{H} \rightarrow \mathcal{H}$ unitary (if $\$$ is Markovian, the possibility of an anti-unitary map is excluded). In Ref. [1] it is argued that the linearity of $\$$, together with the way it is defined (i.e. as a dynamical map from density operators to density operators), implies its complete positivity¹¹. $\$$ would then be a superoperator, which means that it can be made to correspond - via its operator-sum representation - to a unitary mapping in a larger Hilbert space [23, 24]. As will now be shown however, the argument given in Ref. [1] actually only establishes a condition weaker than complete positivity. To this end, consider a system described by a Hilbert space \mathcal{H} and with

⁹The “collapse locality loophole” discussed in the text should not be confused with the so-called “locality loophole”. The latter loophole depends on the fact that it is difficult to experimentally arrange that the events at which the measurement-settings are (randomly and independently) chosen are at spacelike separation from each other and from the source event and it seems to be generally accepted at present that this loophole has effectively been closed [22]. It is also important to note that the collapse locality loophole (in spite of its name) applies equally well to no-collapse interpretations, as may be inferred from the discussion of measurements in the preceding section (the crucial point is whether the act of “property determination”, as defined by the measurement events, occurs in spacelike separated regions).

¹⁰For instance, if one denies the freedom of experimenters to choose the orientation of their measurement apparatuses, the nonlocality demonstrated in Bell-type experiments can be “explained” by arguing that Nature is in an unmitigated conspiracy to consistently give the appearance to behave nonlocally in certain contexts, by denying experimenters the freedom to prove otherwise. But, as has been noted elsewhere [10], it is very difficult to take science seriously upon such a conception of Nature.

¹¹A general linear operator, $T : L(\mathcal{H}) \rightarrow L(\mathcal{H})$, with $L(\mathcal{H})$ denoting the linear space of bounded linear operators on \mathcal{H} , is said to be “completely positive”, if $T \otimes \mathbb{1}$ preserves positivity on $L(\mathcal{H} \otimes \mathcal{H}')$, for any finite-dimensional Hilbert space \mathcal{H}' (i.e. if $T \otimes \mathbb{1}$ applied to any bounded, non-negative linear operator on $\mathcal{H} \otimes \mathcal{H}'$ again yields a bounded, non-negative linear operator, for any finite-dimensional Hilbert space \mathcal{H}'). The definition given in Refs. [23, 24] is more general than the present definition (in the sense that \mathcal{H}' is not restricted to be finite-dimensional and the operators in question are not required to be bounded), whereas the definition in terms of C^* -algebras given in Ref. [25] is slightly more abstract, but is in fact easily seen to include the definition given here. It is then possible to prove [25] that a normal, non-negative linear operator, T , on $L(\mathcal{H})$ is completely positive if and only if it has an operator-sum representation, i.e. if and only if it can be written in the form $T(X) = \sum_i M_i^\dagger X M_i$, for some set $M_i \in L(\mathcal{H})$ and for all $X \in L(\mathcal{H})$ (normality of T basically means that a certain convergence condition on the subspace of self-adjoint operators is preserved by T). The operator-sum representation of a (Markovian) superoperator is used to derive the infinitesimal form of the dynamics, as given by Eq. (9).

time-evolution determined by $\$$, which is entangled with another system, described by a Hilbert space \mathcal{H}' and with no time-evolution. It is then argued in Ref. [1] that the time-evolution of the composite system is given by $\$ \otimes \mathbb{1}$ and that, furthermore, the fact that this map should take any density operator of the composite system into another such operator “is exactly the definition of complete positivity for the map $\$$ [23]”. But the definition of complete positivity given in Refs. [23, 24] requires *every* non-negative operator on $\mathcal{H} \otimes \mathcal{H}'$ to be mapped into another such operator. Without additional information, the argument of Ref. [1] therefore does not establish that $\$$ is necessarily completely positive. However, an initially more serious difficulty with the argument is the fact that time-evolution of the composite system is taken to be represented by $\$ \otimes \mathbb{1}$. The operator $\$$ represents dynamical evolution on $\Xi(\mathcal{H})$ and the operator $\$ \otimes \mathbb{1}$ thus naturally represents dynamical evolution on $\Xi(\mathcal{H} \otimes \mathcal{H}')$ for states of the form $\rho \otimes \rho'$, $\rho \in \Xi(\mathcal{H})$, $\rho' \in \Xi(\mathcal{H}')$. A general state $\sigma \in \Xi(\mathcal{H} \otimes \mathcal{H}')$ is not of this form however and it is therefore not a priori clear that $\$ \otimes \mathbb{1}$ is well-defined on all of $\Xi(\mathcal{H} \otimes \mathcal{H}')$. To this end, it may be noted that a general state, σ , can be written as a sum of products of operators [24], i.e. $\sigma = \sum_i \xi_i \otimes \chi_i$, for certain sets of operators ξ_i, χ_i on \mathcal{H} and \mathcal{H}' , respectively. However, it is not difficult to see that the ξ_i cannot all be simultaneously self-adjoint, non-negative and not traceless for every Hilbert space \mathcal{H}' (if this were the case, taking $\mathcal{H}' = \mathcal{H}$, it would by symmetry also have to be true for the χ_i , but then σ would not be a general state, as any σ -ensemble is a proper mixture). Thus, expressing σ as a sum of products does not make it clear that the action of $\$ \otimes \mathbb{1}$ is well-defined on σ . However, the fact that all σ -ensembles are proper mixtures, implies that it is sufficient to determine whether the action of $\$ \otimes \mathbb{1}$ is well-defined on a general pure state $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}'$. In terms of the Schmidt polar decomposition for $|\Psi\rangle$, i.e. $|\Psi\rangle = \sum_i \sqrt{w_i} |\varrho_i\rangle |\varphi_i\rangle$, with both sets $|\varrho_i\rangle, |\varphi_i\rangle$ orthonormal, this action would be given by

$$\sum_{i,j} \sqrt{w_i w_j} |\varrho_i\rangle \langle \varrho_j| \otimes |\varphi_i\rangle \langle \varphi_j| \longrightarrow \sum_{i,j} \sqrt{w_i w_j} \$ (|\varrho_i\rangle \langle \varrho_j|) \otimes |\varphi_i\rangle \langle \varphi_j| \quad (7)$$

But it is not clear that the right-hand side of this expression is well-defined, since the operators $|\varrho_i\rangle \langle \varrho_j|$ are not elements of $\Xi(\mathcal{H})$ for $i \neq j$ (if a general expansion for $|\Psi\rangle$ is used, the same difficulty is encountered). It is of course conceivable that the domain of definition of $\$$ naturally extends to a larger set, which includes all operators of the form $|\varrho_i\rangle \langle \varrho_j|$ (e.g. the set of all bounded linear operators on \mathcal{H}). For instance, if $\$$ is completely positive, it is clear that the right-hand side of Eq. (7) is formally well-defined, by expressing $\$$ in terms of its operator-sum representation. But it is not necessary for $\$$ to have an operator-sum representation in order for the right-hand side of Eq. (7) to be formally well-defined. For instance, the map $\$_T$, which takes every density operator into its transpose, implements a two-step cyclic time-evolution on $\Xi(\mathcal{H})$ and is such that $\$_T \otimes \mathbb{1}$ is well-defined on all of $\Xi(\mathcal{H} \otimes \mathcal{H}')$ (via the right-hand side of Eq. (7)). But the map $\$_T$ is not completely positive [23] and $\$_T \otimes \mathbb{1}$ does not map $\Xi(\mathcal{H} \otimes \mathcal{H}')$ into itself. However, the point here is that it is conceivable that $\$$ is such that its domain of definition can be naturally extended to include all operators of the form $|\varrho_i\rangle \langle \varrho_j|$ and that $\$ \otimes \mathbb{1}$ maps $\Xi(\mathcal{H} \otimes \mathcal{H}')$ into itself for any finite-dimensional Hilbert space \mathcal{H}' , but that $\$$ is nevertheless not completely positive in the technical sense of the definition, so that it does not have an operator-sum representation. There would then have to be at least one finite-dimensional Hilbert space, \mathcal{H}' , and a bounded, non-negative operator on $\mathcal{H} \otimes \mathcal{H}'$, that is not self-adjoint, and which is mapped by $\$ \otimes \mathbb{1}$ into an operator on $\mathcal{H} \otimes \mathcal{H}'$, that is either not bounded, not non-negative, or both. As far as the present author is able to tell, no mathematical theorem rules out this possibility, although he knows of no example of a map with these properties.

As noted above, if $\$$ is completely positive, it can be made to correspond to a unitary mapping in a larger Hilbert space [23]. This property of $\$$ however does not constrain dynamics to be unitary - or even linear - on the original Hilbert space $\mathcal{H}_I \otimes \mathcal{H}_{II}$ that was used to prove the linearity of $\$$. For instance, on using again the Schmidt polar decomposition for a general state $|\Psi\rangle \in \mathcal{H}_I \otimes \mathcal{H}_{II}$, the following evolution for $|\Psi\rangle := |\Psi(0)\rangle$

$$|\Psi(0)\rangle := \sum_i \sqrt{w_i} |\varrho_i\rangle |\varphi_i\rangle \longrightarrow \sum_i \sqrt{w_i} e^{i w_i \theta t} |\varrho_i\rangle |\varphi_i\rangle =: |\Psi(t)\rangle \quad \theta \in \mathbb{R} \setminus \{0\} \quad (8)$$

is clearly nonlinear, even though the reduced dynamics is linear (and in fact trivial¹²). Furthermore, it was recently demonstrated [28] that a unitary dynamics on a composite Hilbert space in general does not correspond to a reduced dynamics that is completely positive. Since a composite unitary dynamics clearly implies a reduced dynamics of density operators that is linear, this thus confirms the above claim that the linearity of $\$$ (together with the way it is defined) does not imply its complete positivity.

Linearity and complete positivity of $\$$ imply that it is (infinitesimally) given by the Lindblad form¹³ [29]

$$\dot{\rho} = -i[H, \rho] + \sum_j (L_j \rho L_j^\dagger - \frac{1}{2} \{\rho, L_j^\dagger L_j\}) \quad (9)$$

where H represents the Hamiltonian of the system, $\{\cdot, \cdot\}$ denotes the anticommutator and the L_j are so-called quantum jump operators. At the Hilbert space level, the class of evolution equations defined by Eq. (9) includes the nonlinear, stochastic modifications of the Schrödinger equation, that were introduced to solve the measurement paradox [3, 30, 31, 32, 33] (for a recent review, see Ref. [34]). For nonzero L_j , Eq. (9) describes the loss of quantum coherence, i.e. it describes the evolution of pure states into mixed states. Apart from possible applications to the measurement paradox, such evolution has been argued to be relevant to describe the process of black hole evaporation [2, 35, 36] (for an astrophysical black hole such evaporation should occur as soon as its Hawking temperature drops below the temperature of the cosmic background radiation). The loss of quantum coherence is often regarded as something unphysical, since it is expected to give rise to violations of either causality or energy-momentum conservation [37]. It has been shown however [38] that it is possible to define a class of Markovian models in which pure states evolve into mixed states, but which do not lead to pathological behaviour at scales accessible to ordinary laboratory physics. In addition, the argument of Ref. [37] is based on the assumption that the process of black hole formation and evaporation is effectively modeled by the evolution (9), whereas it is unclear that the Markovian character of this evolution is physically reasonable in the case of black holes [38].

Finally, it is important to note that the argument for linear dynamics does not continue to be valid in contexts that go beyond standard quantum kinematics. For instance, Weinberg [39] has proposed a nonlinear generalization of quantum theory, in which observables are represented by real-valued functions on \mathcal{H} , homogeneous of degree one in Ψ , $\bar{\Psi} \in \mathcal{H}$ (where Ψ and $\bar{\Psi}$ are taken as independent variables). Within this framework, the observables of standard quantum theory are represented by the subset of bilinear, real-valued functions - corresponding to expectation value functions of self-adjoint linear operators. This naturally suggests a generalization of the notion of a state as a general probability distribution, ρ , on “quantum phase space”, i.e. the projective Hilbert space, $P\mathcal{H}$ (where it is recalled that $P\mathcal{H}$ is equipped with a symplectic structure that is generated by the imaginary part of the inner product on \mathcal{H}). The trace rule for the expectation value of a standard quantum observable, which generalizes to the value of a general Weinberg-observable for pure states, then naturally extends to a ρ -weighted integral over $P\mathcal{H}$ of a Weinberg-observable for mixed states. On such an account, the density operator, $\hat{\rho}$, of ordinary quantum theory is only the first moment of a one-dimensional projection operator in the state ρ (i.e. the ρ -weighted integral of a one-dimensional projection operator over $P\mathcal{H}$), which implies that $\hat{\rho}$ does not in general contain all the information about a system, when dealing with nonlinear observables [40]. Failure to take this fact into account can lead to contradictory results [41], although it is not clear that a proper treatment of the quantum state guarantees the absence of such results.

¹²This example is taken from Ref. [26]. For a different example, see Ref. [27].

¹³Strictly speaking it is also necessary to impose the condition that $\$$ be of a Markovian nature. This has been assumed (implicitly) throughout the discussion.

4 Unentangled Systems : Linear Dynamics and Preservation of Mixture Independence

As already remarked in the introduction, a different argument for the linearity of quantum dynamics was recently advanced by Jordan [5]. He considers the dynamics for a system which is part of a larger system, but only so in a classical sense. That is, the system, which is denoted by S in Ref. [5], is combined with another system, denoted by R, but there is no entanglement between S and R. As will become clear in the following, these specific assumptions are somewhat unfortunate, as they tend to deflect attention away from the core of the argument. In order to bring it into accordance with the notation and language used in the previous two sections, Jordan's argument is first reformulated and slightly generalized.

Given any density operator, ρ , for S and any ρ -ensemble, $\{w_i, P_{\psi_i}\}$, Jordan considers two possible density operators for the combined system that both have ρ as a reduced density operator

$$\bar{\Pi}_{\{w, P_{\psi}\}} := \sum_i w_i P_{\psi_i} \otimes P_{\alpha_i} \quad \Pi_{\rho} := \rho \otimes \sum_i s_i P_{\alpha_i} \quad (10)$$

with $P_{\alpha_i} := |\alpha_i\rangle\langle\alpha_i|$ time-independent, orthogonal projectors for R, $0 < s_i < 1$ and $\sum_i s_i = 1$ (cf. Eqs. (2.4), (2.7) respectively in Ref. [5]). The interpretation of these density operators will be addressed shortly, something which is in fact crucial for the entire argument. The key assumption made by Jordan is that the dynamics of S does not depend on how S is "embedded" within the larger system. He also assumes that the operator implementing time-evolution over some fixed time-interval is defined both on ρ directly and on the components P_{ψ_i} of the particular decomposition $\{w_i, P_{\psi_i}\}$ of ρ . On denoting this operator again by $\$$, it then immediately follows from Jordan's key assumption that

$$\$\rho = \$ \sum_i w_i P_{\psi_i} = \sum_i w_i \$P_{\psi_i} \quad (11)$$

so that $\$$ is indeed linear.

The two above assumptions will now be questioned, by addressing the physical interpretation of the two density operators $\bar{\Pi}$ and Π . As for $\bar{\Pi}$, measurements on R that distinguish between the orthonormal states $|\alpha_i\rangle$, generate the ρ -ensemble $\{w_i, P_{\psi_i}\}$. If the same measurement procedures for R are applied to Π however, it seems at first that the most one can say for S is that its state is given by ρ , regardless of which state the system R is found to be in. But since there is no entanglement, all ρ -ensembles are proper mixtures, and it is therefore possible to consistently think of the system S as actually being in some definite state in \mathcal{H}_S with a certain classical probability. Hence, in the situation where the two states are prepared experimentally, in the case of Π , a fraction $\tilde{w}_i s_j$ of a large number of combined systems is definitely prepared in the state $P_{\tilde{\psi}_i} \otimes P_{\alpha_j}$, with $\{\tilde{w}_i, P_{\tilde{\psi}_i}\}$ denoting any ρ -ensemble, whereas in the case of $\bar{\Pi}$, a fraction w_i of the combined systems is prepared in the state $P_{\psi_i} \otimes P_{\alpha_i}$. Upon an ignorance interpretation of $\bar{\Pi}$ and Π , the difference that remains between the two states is that there is no one-to-one correspondence between physically realized states in \mathcal{H}_S and \mathcal{H}_R in the latter case. With the ensembles realized physically, what evolves with time are the individual pure states, P_{ψ_i} , $P_{\tilde{\psi}_i}$, so that $\$$ is initially only defined on the set of all such states. The time-evolved ensembles are then represented by

$$\sum_i w_i \$P_{\psi_i} \otimes P_{\alpha_i} \quad \text{and} \quad \sum_{i,j} \tilde{w}_i s_j \$P_{\tilde{\psi}_i} \otimes P_{\alpha_j} \quad (12)$$

for respectively $\bar{\Pi}$ and Π . Jordan's key assumption is that these two different evolved states for the combined system have to result in the same state for S, which immediately implies the equation analogous to Eq. (2) for the current situation. In other words, the possibility of different statistical mixtures corresponding to the same density operator evolving into statistical mixtures corresponding to different density operators is ruled out by assumption. As before, this assumption immediately implies that $\$$ can be extended to a well-defined map on $\Xi(\mathcal{H}_S)$, which is linear in

the sense of Eq. (11).

Thus, to summarize, because there is no entanglement, the assumption that $\$$ has to act directly on ρ - as apparently suggested by the form (10) for Π - is found to be unjustified. In a concrete experimental situation, a particular ρ -ensemble is realized physically and it are the pure states in such an ensemble on which $\$$ acts.

Let us now address Jordan's key assumption. At first, it appears to be very reasonable on physical grounds, as there is no interaction between the systems S and R. On closer inspection however, the issue of whether the assumption is physically reasonable or not, has nothing to do with the absence of interactions between S and R, but instead turns out to crucially depend on a peculiar property of density operators in ordinary quantum theory. An experimenter confronted with a large ensemble of physical systems of which a fraction w_i was prepared in the pure state $|\psi_i\rangle$, cannot ascertain this fact according to ordinary quantum theory. All he can know by doing experiments on the ensemble is the density operator, ρ , determined by this particular ensemble. Thus, in standard quantum theory it is possible that ensembles of systems, which were experimentally prepared *differently*, are represented by the *same* physical state. In fact, what was said before shows that the statement that density operators are subject to a linear dynamics is *equivalent* to requiring that this property of ensemble-independence of states is preserved by the dynamics. But, is it physically reasonable to impose such a requirement ? From experience with ordinary quantum theory one is tempted to think that it is, but if one is to *derive* the linearity of quantum dynamics, it is necessary to give a different motivation. In fact, it does not seem that there is a good physical justification - apart from causality and expectations based on experience with ordinary quantum theory - for requiring that differently prepared ensembles corresponding to the same density operator always evolve into ensembles that again correspond to a single density operator. Indeed, with a dynamics that physically distinguishes different mixtures belonging to the same initial density operator, it becomes necessary to sharpen the interpretational rules of ordinary quantum theory [19, 42]. It is of course true that a nonlinear dynamics of density operators appears highly unnatural within the linear context of standard quantum kinematics (for instance, on $\Xi(\mathcal{H}_S)$, such dynamics would be represented by bifurcating dynamical trajectories) and that the causality argument of section 2 moreover provides a strong, independent reason for rejecting such a nonlinear dynamics. But, as already stated in the introduction, the possibility is contemplated of a generalized dynamics with standard kinematics being a lowest-order approximation to a deeper nonlinear theory, and the aim in this note was to investigate the extent to which dynamics is constrained by the kinematics.

5 Conclusion

To recapitulate the logical structure of the two arguments for linear quantum dynamics; both arguments attempt to derive linearity from certain conditions, which are easily recognized as in fact being *implied* by linearity themselves. In the first argument this condition is causality, which may be viewed as a primitive physical condition, one that must be effectively satisfied in order to avoid actual pathologies. It was seen that for causality to imply linearity, it seems to be necessary to assume that measurements are accompanied by objective state reduction, something which is consistent with arguments for linear dynamics put forward by earlier authors. It was also seen that this actually leads to a loophole in the argument, which will presumably be closed by future experiments. In the view of this author, the strength of this way of arguing for linear dynamics is that it relates two concepts which are a priori unrelated, one of which may be regarded as a physical axiom. In the second argument, the condition implied by linear dynamics is that different proper mixtures belonging to the same initial density operator are not physically distinguished by the dynamics. It was argued however, that such a condition is not a very reasonable one to impose, if generalizations of the standard quantum dynamics are contemplated.

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